Abstract—In this paper we derive theoretical results for the problem of on-line odometry self-calibration for a mobile robot. A first series of results regards the problem of understanding if a given system (consisting of a robot with several sensors) contains the necessary information to perform the on-line self calibration of the odometry. We consider several cases corresponding to different odometry systems and different types of robot sensors. Finally, we also consider the problem of maximizing the calibration accuracy and we formulate this problem as an optimal control problem. For the special case of a holonomic vehicle, we derive second order differential equations characterizing the solution, i.e. defining the best trajectory which maximizes the calibration accuracy.

I. INTRODUCTION

Calibration is the problem of estimating the parameters characterizing the systematic error of a sensor. In mobile robotics, performing this process on-line is not only a desire which automatizes a work which would have to be performed by hand, but it is in many cases a real need for application-like scenarios. This is especially true for the odometry. Indeed, the pressure of tires can change over time and the effective wheelbase depends on the terrain where the robot is moving. Having a system able to adapt continuously to different floor types and changing wheels attributes (i.e. different tire pressure, deterioration, etc.) is a key advantage.

Several strategies have been developed to perform on-line self-calibration. In order to do this, the mobile robot is equipped with at least one other sensor. In many cases, an Extended Kalman Filter has been introduced to simultaneously estimate the robot configuration and the parameters characterizing the systematic error of the odometry sensor (i.e., to solve the Simultaneous Localization and Auto Calibration (SLAC) problem). Regarding the odometry, the SLAC problem has been investigated in [3], [4], [7], [8], [12] and [13] both for indoor and outdoor environments. Very recently, the same idea was adopted to self calibrate a vision sensor [9].

Although in most cases the strategies proposed perform well, the following two questions remain open:

• Given a system consisting of a mobile robot with several sensors, does the system contain the necessary information to perform the calibration of the odometry sensor and/or to perform SLAC?

• What is the best robot trajectory which maximizes the calibration accuracy?

An answer to the second question for a mobile robot with differential drive could generalize the off-line method usually adopted to calibrate the odometry, the UMBmark [1]. This method requires that the robot moves along a square path in both clockwise and counterclockwise direction.

Very recently, we answered the first question for non-holonomic vehicles equipped with several types of exteroceptive sensors [10]. Furthermore, in [10] we answered the second question for holonomic vehicles in the case when the final robot configuration is not assigned. In this paper we give an answer to the previous two questions for other systems. In section II we define these systems. In section III we answer the first question both for holonomic and non-holonomic vehicles. In section IV, we answer the second question for the case of holonomic vehicles. With respect to [10] we take into account the additional constraint due to an assigned final robot configuration. In particular, we derive second order differential equations characterizing the solution. Finally, conclusions are presented in section V.

II. THE SYSTEMS

We consider two types of mobile robot. The former is a non-holonomic vehicle equipped with a differential drive system, the latter is a holonomic vehicle. We also consider several systems, corresponding to the previous two types of robot equipped with different exteroceptive sensors. We introduce the systems separately for each type of robot.

A. Non-holonomic Vehicle

The configuration of the robot in a global reference frame can be characterized through the vector \( X = [x_R, y_R, \theta_R]^T \) where \( x_R \) and \( y_R \) are the cartesian robot coordinates and \( \theta_R \) is the robot orientation. The dynamics of this vector are described by the following non-linear differential equations:

\[
\dot{X} = f(X, \dot{u}) = \begin{bmatrix}
\dot{x}_R = v \cos \theta_R \\
\dot{y}_R = v \sin \theta_R \\
\dot{\theta}_R = \omega
\end{bmatrix}
\]  

(1)

where \( v \) and \( \omega \) are the linear and the rotational robot speed, respectively. The link between these velocities and the robot controls \( \dot{u} \) depends on the considered robot system drive. We will consider the case of a differential drive. In order
to characterize the systematic odometry error we adopt the model introduced in [1]. We have:
\[
v = \frac{\delta_R \hat{v}_R + \delta_L \hat{v}_L}{2} \quad \omega = \frac{\delta_R \hat{v}_R - \delta_L \hat{v}_L}{\delta b}
\]
where \( \hat{v}_R \) and \( \hat{v}_L \) are the control velocities (i.e. \( \hat{u} = [\hat{v}_R, \hat{v}_L]^T \)) for the right and the left wheel, \( b \) is the nominal value for the distance between the robot wheels and \( \delta_R, \delta_L \) and \( \delta_b \) characterize the systematic odometry error due to an uncertainty on the diameters of the wheels and on the distance between the wheels. The equation (2) does not take into account the non-systematic odometry error.

Our robot is also equipped with one or several exteroceptive sensors able to provide observation \( y = h(X) \) where \( h \) is the observation function. We assume that a landmark exists in our environment and, without loss of generality, we fix the global reference frame on it. We consider separately the following four observations (see figure 1 for an illustration):

1) the distance of the landmark \( (D = \sqrt{x_R^2 + y_R^2}) \)
2) the bearing of the landmark \( (\beta = \arctan 2(y_R, x_R) - \pi) \)
3) both \( D \) and \( \beta \)
4) both \( D \) and the absolute robot orientation \( (\theta_R) \).

These observations define four different systems which will be considered in the following separately.

**B. Holonomic Vehicle**

The configuration of an omnidirectional vehicle can be characterized through the state \( X = [x_R, y_R]^T \) whose dynamics are:
\[
\dot{X} = f(X, \hat{u}) = \begin{bmatrix} \dot{x}_R = \delta_x \hat{v}_x \\ \dot{y}_R = \delta_y \hat{v}_y \end{bmatrix}
\]
where \( \hat{v}_x \) and \( \hat{v}_y \) are the control velocities (i.e. \( \hat{u} = [\hat{v}_x, \hat{v}_y]^T \)) and \( \delta_x \) and \( \delta_y \) characterize the systematic odometry error. The equation (3) does not take into account the non-systematic odometry error.

As in the previous case, we consider a landmark in the environment and we introduce two additional systems corresponding to the following two observations:

1) the distance of the landmark \( (D = \sqrt{x_R^2 + y_R^2}) \)
2) the bearing of the landmark \( (\beta = \arctan 2(y_R, x_R) - \pi) \)

**III. Observability Properties**

In the previous section we introduced six different robotic systems which are defined by the drive system (non-holonomic differential and holonomic) and by an observation (four for the non-holonomic vehicle and two for the holonomic one). In this section we want to investigate for each system the observability properties of the parameters characterizing the systematic odometry error (\( \delta_R, \delta_L \) and \( \delta_b \) for the differential odometry error and \( \delta_x \) and \( \delta_y \) for the holonomic vehicle). In other words, we want to understand in which systems the calibration of the odometry can be performed. Furthermore, we want to investigate the observability properties of the robot configuration, namely we want to understand when the calibration can be performed simultaneously with the robot localization (the SLAC problem). In order to answer these questions we carry out an observability analysis which takes into account the system nonlinearities. We apply the observability rank condition as introduced by Hermann and Krener in [6] to determine if the systems defined through the four observations previously introduced have the local distinguishability property. We remark that our systems are affine in the input variables, i.e. the function \( f \) in (1) and (3) has the following structure:
\[
f(X, \hat{u}) = \sum_{k=1}^M f_k(X) \hat{u}_k
\]
where \( M \) is the number of the control inputs (\( M = 2 \) in both cases).

We now want to remind some theoretical concepts by Hermann and Krener in [6]. We will adopt the following notation. We indicate the \( K^{th} \) order Lie derivative of a field \( \psi \) along the vector fields \( v_{i1}, v_{i2}, ..., v_{iM} \) with \( L^K v_{i1}, v_{i2}, ..., v_{iM} \psi \). Note that the Lie derivative is not commutative. In particular, in \( L^K v_{i1}, v_{i2}, ..., v_{iM} \psi \) it is assumed to differentiate along \( v_{i1} \) first and along \( v_{iK} \) at the end. Let us indicate with \( \Omega \) the space spanned by all the Lie derivatives \( L^K v_{i1}, v_{i2}, ..., v_{iM} \psi \). Furthermore, we denote with \( d\Omega \) the space spanned by the gradients of the elements of \( \Omega \). In this notation, the observability rank condition can be expressed in the following way: The dimension of the part of the system which is observable at a given \( X_0 \) is equal to the dimension of \( d\Omega \).

In the rest of this section we carry out the observability analysis separately for the holonomic and non-holonomic vehicle.
A. Non-holonomic Vehicle with Differential Drive

We found that the computation becomes significantly easier if, for the robot position, we adopt polar coordinates instead of cartesian ones. In these coordinates, the robot configuration is \( R = [D, \chi, \theta_R]^T \) with \( x_R = D \cos \chi \) and \( y_R = D \sin \chi \). From (1) and (2) we have:

\[
\begin{bmatrix}
\dot{D} = A(\dot{v}_R + B\dot{v}_L) \cos \theta_R \\
\dot{\chi} = \frac{A(\dot{v}_R + B\dot{v}_L)}{D} \sin \theta_R \\
\dot{\theta}_R = C(\dot{v}_R - B\dot{v}_L) \\
\dot{A} = \dot{B} = \dot{C} = 0
\end{bmatrix}
\]

where \( A = \frac{\delta_a}{2}, B = \frac{\delta_r}{r} \) and \( C = \frac{1}{2} \frac{\delta_a}{r} \). In the case the goal is to perform only the calibration (i.e. we are not interested in estimating also the robot configuration) we can consider the dynamics of \( D \) and \( \theta = \theta_R - \chi \):

\[
\begin{bmatrix}
\dot{D} = A(\dot{v}_R + B\dot{v}_L) \cos \theta \\
\dot{\theta} = C(\dot{v}_R - B\dot{v}_L) - \frac{A(\dot{v}_R + B\dot{v}_L)}{D} \sin \theta \\
A = B = C = 0
\end{bmatrix}
\]

1) Distance: Let us start by considering the case of the observation \( h(X) = D = \sqrt{x_R^2 + y_R^2} \). We investigate the case of calibration and therefore we refer to the dynamics in (6). By comparing with (4) we get:

\[
\begin{align*}
\dot{f}_1 &= \left[ A \cos \theta, C - \frac{A}{D} \sin \theta, 0, 0 \right]^T \\
\dot{f}_2 &= \left[ AB \cos \theta, -BC - \frac{AB}{D} \sin \theta, 0, 0 \right]^T
\end{align*}
\]

By a direct computation (see the appendix) it is possible to prove the independence of the gradients of the following Lie derivatives: \( L_0 h, L_1 h, L_2 h, L_3 h \) and \( L_{11} h \). This means that the dimension of \( d\Omega \) is 5 and therefore the state \([D, \theta, A, B, C]^T\) can be observed. Hence the system contains the necessary information to perform the self-calibration.

2) Bearing: Let us consider now the case of the bearing observation \( (\beta = \pi - \theta) \). By defining \( D_a = \frac{D}{2} \) we get from the equation (6)

\[
\begin{bmatrix}
\dot{D}_a = (\dot{v}_R + B\dot{v}_L) \cos \theta \\
\dot{\theta} = C(\dot{v}_R - B\dot{v}_L) - \frac{(\dot{v}_R + B\dot{v}_L)}{D_a} \sin \theta \\
\dot{A} = \dot{B} = \dot{C} = 0
\end{bmatrix}
\]

The observation \( \beta \) does not provide any information to distinguish \( A \) from \( D \). In other words, the best we can hope is that the state \([D_a, \theta, B, C]^T\) is observable.

By comparing (8) with (4) we get:

\[
\begin{align*}
\dot{f}_1 &= \left[ \cos \theta, C - \frac{\sin \theta}{D}, 0, 0 \right]^T \\
\dot{f}_2 &= \left[ B \cos \theta, -BC - \frac{B}{D} \sin \theta, 0, 0 \right]^T
\end{align*}
\]

By a direct computation (as in the previous case) it is possible to prove the independence of the gradients of the following Lie derivatives: \( L^0 h, L^1 h, L^2 h, L^3 h \) and \( L^{11} h \). This means that the dimension of \( d\Omega \) is 4 and therefore the state \([D_a, \theta, B, C]^T\) can be observed.

3) Bearing and Distance: The system contains the information to perform the self-calibration as proven in III-A.1. We want to see if at the same time the robot configuration \([D, \chi, \theta_R]^T\) is observable. In other words, we want to see if it is possible to perform SLAC. We will prove that it is not possible. Indeed, from the equation (5), the vector field \( f_i \) defined in (4) will depend on \( \theta_R \) and \( \chi \) only through the difference \( \theta_R - \chi \). On the other hand, also \( \beta \) depends on \( \theta_R \) and \( \chi \) through this difference. Finally, \( D \) is independent of them. This means that all the Lie derivatives will have gradients with the following structure:

\[
[q_1, q_2, q_3, q_4, q_5, q_6]^T
\]

with \( q_2 = -q_3 \). Therefore, the dimension of \( d\Omega \) will be less than or equal to 5.

4) Distance and Absolute Orientation: In this case the observation consists of \( h_D = D \) and \( h_{\theta_R} = \theta_R \). By comparing (5) with (4) we get:

\[
\begin{align*}
f_1 &= \left[ A \cos \theta, A \frac{\sin \theta}{D}, C, 0, 0, 0 \right]^T \\
f_2 &= \left[ AB \cos \theta, AB \frac{\sin \theta}{D}, -BC, 0, 0, 0 \right]^T
\end{align*}
\]

By a direct computation it is possible to prove the independence of the gradients of the following Lie derivatives: \( L^0 h_{\theta_R}, L^1 h_{\theta_R}, L^2 h_{\theta_R}, L^3 h_D, L^3 h_D \) and \( L^{11} h_D \). This means that the dimension of \( d\Omega \) is 6 and therefore it is possible to perform SLAC.

We summarize the results of this section through the following theorems.

**Theorem 1** It is possible to self-calibrate the odometry when the robot is equipped with a sensor able to provide its distance from a single landmark in the environment.

**Theorem 2** It is possible to partially self-calibrate the odometry when the robot is equipped with a sensor able to provide the bearing of a single landmark in the environment. In particular, the calibration can be performed up to a scale factor.

**Theorem 3** It is not possible to perform SLAC when the observation consists of the bearing and the distance of a single landmark in the environment.

**Theorem 4** It is possible to perform SLAC when the observation consists of the distance of a single landmark in the environment and the absolute robot orientation.
B. Holonomic Vehicle

In this case we only consider the case of SLAC. The state we want to estimate is $X = [x_R, y_R, \delta_x, \delta_y]^T$ whose dynamics are:

$$
\begin{bmatrix}
\dot{x}_R \\
\dot{y}_R \\
\dot{\delta}_x \\
\dot{\delta}_y
\end{bmatrix} =
\begin{bmatrix}
\delta_x \dot{v}_x \\
\delta_y \dot{v}_y \\
0 \\
0
\end{bmatrix}
$$

(11)

1) Distance: For the sake of simplicity, instead of the distance we consider the observation $h(X) = \frac{D^2}{2} = \frac{x_R^2 + y_R^2}{2}$, containing the same information. By comparing (11) with (4) we get:

$$
\begin{align*}
& f_1 = [\delta_x, 0, 0, 0]^T \\
& f_2 = [0, \delta_y, 0, 0]^T
\end{align*}
$$

(12)

By a direct computation it is possible to prove the independence of the gradients of the following Lie derivatives: $L^0h$, $L^1h$, $L^2h$, $L^3h$. This means that the dimension of $d\Omega$ is 4 and therefore the state $[x_R, y_R, \delta_x, \delta_y]^T$ can be observed. Hence the system contains the necessary information to perform SLAC.

2) Bearing: In this case the observation depends on $x_R$ and $y_R$ only through the $\text{atan}$ function ($\beta = h(X) = \text{atan2}(y_R, x_R) - \pi$). This means that it depends on the ratio $y_R/x_R$.

By dividing the equations in (11) by one of the $\delta$ parameters (e.g. $\delta_x$) we get:

$$
\begin{bmatrix}
\dot{x}_d \\
\dot{y}_d \\
\dot{\delta}
\end{bmatrix} =
\begin{bmatrix}
\dot{v}_x \\
\delta \dot{v}_y \\
0
\end{bmatrix}
$$

(13)

where $x_d = \frac{x_R}{\delta_x}$, $y_d = \frac{y_R}{\delta_x}$ and $\delta = \frac{\delta_x}{\delta_x}$. The observation $\beta$ does not provide any information to distinguish $x_R$ and $y_R$ from $\delta$. In other words, the best we can hope is that the state $[x_d, y_d, \delta]^T$ is observable.

By comparing (13) with (4) we get:

$$
\begin{align*}
& f_1 = [1, 0, 0]^T \\
& f_2 = [0, 0, 0]^T
\end{align*}
$$

(14)

By a direct computation it is possible to prove the independence of the gradients of the following Lie derivatives: $L^0h$, $L^1h$, $L^2h$. This means that the dimension of $d\Omega$ is 3 and therefore the state $[x_d, y_d, \delta]^T$ can be observed.

We summarize the results of this section through the following theorems.

Theorem 5 It is possible to perform SLAC when the observation consists of the distance of a single landmark in the environment.

Theorem 6 It is possible to perform SLAC up to a scale factor when the observation consists of the bearing of a single landmark in the environment.

IV. Optimal Trajectories for Self-Calibration

Once we know that a given system contains the necessary information to perform the estimation of a set of parameters, the fundamental question arising is: what is the best choice for the control input in order to minimize the error in the estimation of these parameters?

In order to answer the previous question it is necessary to specify first of all the method which is adopted to carry out the estimation. Indeed, depending on the method, we have a precise characterization for the estimation error. In this paper we consider the case of the Extended Kalman Filter (EKF) since it has been widely used in previous works, both for SLAM and SLAC. In this case the characterization of the estimation error is provided by a covariance matrix $P$ satisfying the Riccati differential equation [2]. This characterization can be adopted in a real implementation (where the system is actually time-discrete) since, starting from a discrete-time system, it is possible to derive an equivalent continuous-time system model as shown in [11]. The structure of the Riccati equation is:

$$
\frac{dP}{dt} = FP + PF^T + Q - fPH^TR^{-1}HP
$$

(15)

where $F$ is the Jacobian of the dynamics with respect to the state to be estimated, $Q$ characterizes the noise in the dynamics, $R$ characterizes the observation error, $H$ is the Jacobian of the observation with respect to the state to be estimated, $f$ is the observation frequency.

Once we have a characterization for the estimation error, we must specify all the constraints of the problem. These constraints can be fixed (or a maximum) length of the trajectory, a fixed initial and final robot configuration, a fixed (or a maximum) length of time to perform the estimation, a fixed (or a maximum) cost in terms of energy and special requirements on the control input.

We remark that for all the systems considered in section III-A the solution of the Riccati differential equation can be obtained only numerically.

In the following, we will restrict our analysis to the holonomic vehicle introduced in II-B. Furthermore, we consider the case of the distance observation, since in this case the system contains the information to perform the estimation (Theorem 5). We characterize the observation by assuming a Gaussian error with variance $\sigma^2_{\text{obs}}$ and a frequency $f$. Furthermore, we assume that the energy cost depends quadratically on the vehicle speed (this is a suitable hypothesis for many cases). Finally, the solution has to satisfy the following requirements:

1) The length of time allowed for this estimation has to be equal to a given value $T$. Since we assumed a constant frequency $f$ for the observations, this means that we fix the number of observations.

2) The total energy cost has to be equal to a fixed value, i.e. $\int_0^T (\dot{x}^2 + \dot{y}^2) \, dt = E$, where $E$ is the allowed energy cost and $\lambda$ is a parameter which depends on the environment and the vehicle.
3) The initial and the final robot position (i.e., the position at the time 0 and T) are assigned.

The initial covariance matrix for the state $[x_R, \delta_x, y_R, \delta_y]^T$ characterizing the uncertainty on the state at the initial time, is

$$P(0) = P_0 = \begin{bmatrix} \sigma_x^2 & 0 & 0 & 0 \\ 0 & \sigma_x^2 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & 0 \\ 0 & 0 & 0 & \sigma_y^2 \end{bmatrix}$$

(16)

We look for the solution of the Riccati differential equation by setting $P = UV^{-1}$ (see [2] for the details). In these new variables the Riccati equation becomes:

$$\begin{bmatrix} \dot{U} \\ \dot{V} \end{bmatrix} = FU + V^T \dot{U} + CU$$

(17)

where

$$F = \begin{bmatrix} F_x & 0_{2 \times 2} \\ 0_{2 \times 2} & F_y \end{bmatrix} \quad F_{x/y} = \begin{bmatrix} 0 & \hat{\dot{\nu}}_{x/y} \\ 0 & 0 \end{bmatrix}$$

(18)

and

$$C = f\sigma_{obs}^{-2} \begin{bmatrix} C_x \\ C_m \\ C_y \end{bmatrix}$$

(19)

where

$$C_x = \begin{bmatrix} x_R & 0 \\ 0 & 0 \end{bmatrix} \quad C_y = \begin{bmatrix} 0 & y_R \\ 0 & 0 \end{bmatrix}$$

$$C_m = \begin{bmatrix} x_R & y_R \\ 0 & 0 \end{bmatrix}$$

(20)

In order to have $P(0) = P_0$ we set the following initial conditions on $U$ and $V$: $U(0) = P_0$ and $V(0) = I$, where $I$ is the identity matrix $4 \times 4$. The solution of the first equation in (17) is

$$U = \begin{bmatrix} U_x & 0_{2 \times 2} \\ 0_{2 \times 2} & U_y \end{bmatrix}$$

(21)

$$U_{x/y} = \begin{bmatrix} \sigma_{x/y}^2 & \sigma_{x/y} \hat{\dot{\nu}}_{x/y}(t) \\ 0 & \sigma_{x/y}^2 \end{bmatrix}$$

(22)

By directly integrating the equations in (11) we get:

$$x_R(t) = x_0 + \delta_x \hat{\dot{\nu}}_x(t) \approx x_0 + \hat{\dot{\nu}}_x(t)$$

$$y_R(t) = y_0 + \delta_y \hat{\dot{\nu}}_y(t) \approx y_0 + \hat{\dot{\nu}}_y(t)$$

(23a)

(23b)

since we assume $\delta_x/y \approx 1$. To proceed, we write the matrix $V$ in the following block form:

$$V = \begin{bmatrix} V_x & V_{xy} \\ V_{yx} & V_y \end{bmatrix}$$

(24)

where each block is $2 \times 2$. From (17), (18), (19) and (21) we get:

$$V_{x/y} = -F_{x/y}^T V_{x/y} + kC_{x/y} U_{x/y}$$

(25)

where $k = f\sigma_{obs}^{-2}$. Our goal is to minimize the matrix $P(T) = U(T)V(T)^{-1}$. For the sake of simplicity, we maximize the inverse of this matrix, i.e. $P(T)^{-1} = V(T)U(T)^{-1}$.

We adopt as criterion of matrix maximization, the trace maximization. Therefore it suffices to compute the two blocks $V_x U_{x}^{-1}$ and $V_y U_{y}^{-1}$ at the time $T$. In particular,

$$tr(P^{-1}) = tr(V_x U_{x}^{-1}) + tr(V_y U_{y}^{-1})$$

(26)

From (21) we have:

$$U_{x/y}^{-1} = \begin{bmatrix} \sigma_{x/y}^{-2} & -\hat{\dot{\nu}}_{x/y}(t) \sigma_{x/y}^{-2} \\ 0 & \sigma_{x/y}^{-2} \end{bmatrix}$$

(27)

From (25) it is clear that the computation can be carried out separately for the $x$ and $y$ dimension. Furthermore, this computation is the same for the two directions. Therefore, we consider only the case of $x$ and for simplicity we do not put the pedix $x$. Therefore, $V_{11}$ will indicate $V_{x,x}$.

From (18), (19), (20), (21), (23) and (25) we obtain:

$$\begin{align*}
V_{11} &= k(x_0 + \hat{\dot{\nu}})^2\sigma^2 \\
V_{12} &= k\sigma_\delta^2 \left( x_0^2 \hat{\Omega} + 2x_0 \hat{\Omega}_2 + \hat{\Omega}_3 \right)
\end{align*}$$

(28a)

(28b)

whose solutions are:

$$\begin{align*}
\hat{\Omega}_1(t) &= \int_0^t \hat{\dot{\nu}}(\tau)d\tau, \\
\hat{\Omega}_2(t) &= \int_0^t \hat{\dot{\nu}}^2(\tau)d\tau, \\
\hat{\Omega}_3(t) &= \int_0^t \hat{\dot{\nu}}^3(\tau)d\tau
\end{align*}$$

(29)

(30)

(31a)

(31b)

(31c)

From (25), (29) and (30) we obtain:

$$\begin{align*}
V_{21} &= 1 + k\sigma^2 \left[ x_0^2 + 2x_0 \left( \hat{\Omega}_2 - \hat{\Omega}_3 \right) + \hat{\Omega}_3 - \hat{\Omega}_2 \hat{\Omega}_2 \right] \\
V_{22} &= 1 + k\sigma_\delta^2 \left[ x_0^2 + 2x_0 \left( \hat{\Omega}_3 - \hat{\Omega}_2 \hat{\Omega}_2 \right) + \hat{\Omega}_4 - \hat{\Omega}_3 \hat{\Omega}_2 \right]
\end{align*}$$

(32)

(33)

From (26) and (27) we obtain for the trace of $P^{-1}$ at the time $T$:

$$tr(P^{-1}) \equiv -S[\hat{\dot{\nu}}_x, \hat{\dot{\nu}}_y] = V_{x11}(T) - \hat{\dot{\nu}}_x(T) V_{x12}(T) + V_{x22}(T)$$

$$+ V_{y11}(T) - \hat{\dot{\nu}}_y(T) V_{y12}(T) + V_{y22}(T)$$

(34)

We remark that both $\hat{\dot{\nu}}_x(T)$ and $\hat{\dot{\nu}}_y(T)$ are fixed since the final robot position is assigned.

$$\begin{align*}
\hat{\dot{\nu}}_x(T) &= x_f - x_0 \\
\hat{\dot{\nu}}_y(T) &= y_f - y_0
\end{align*}$$

(35)
Furthermore, from (22)
\[ \ddot{w}_x(0) = \ddot{w}_y(0) = 0 \] (36)

Our goal is to find the functions  \( \ddot{w}_x \) and  \( \ddot{w}_y \) which minimize the functional  \( S \) in (34) (remark the minus sign introduced to have a minimization instead of a maximization). Furthermore, the solutions have to satisfy (35) and (36). From the expressions (29), (32) and (33), the functional  \( S \) can be expressed as follows:
\[ S[\ddot{w}_x, \ddot{w}_y] = \int_0^T \{ L_x(\ddot{w}_x(t)) + L_y(\ddot{w}_y(t)) \} dt \] (37)

where both  \( L_x \) and  \( L_y \) are polynomials of fourth order in  \( \ddot{w}_x \) and  \( \ddot{w}_y \), respectively. Finally, our solution has to satisfy the energy constraint:
\[ \int_0^T \left\{ \ddot{w}_x(t)^2 + \ddot{w}_y(t)^2 \right\} dt = \frac{E}{\lambda} \equiv E_\lambda \] (38)

therefore, by introducing the Lagrange multiplier  \( \mu \) to take into account (38) we have to find the extrema of the following functional:
\[ S_\mu[\ddot{w}_x, \ddot{w}_y] = \mu \int_0^T \left\{ \ddot{w}_x(t)^2 + \ddot{w}_y(t)^2 \right\} dt + S[\ddot{w}_x, \ddot{w}_y] \] (39)

This is the action of a particle of mass  \( 2\mu \) in the potential field  \( \phi = -L_x(\ddot{w}_x) - L_y(\ddot{w}_y) \) [5]. Therefore, the solution satisfies the Euler-Lagrange differential equations:
\[ \mu \ddot{w}_x = \frac{1}{2} \frac{\partial L_x}{\partial w_x} \] (40a)
\[ \mu \ddot{w}_y = \frac{1}{2} \frac{\partial L_y}{\partial w_y} \] (40b)

Since they are two independent second order differential equations, the solutions depend on four parameters. Furthermore, both equations depend on the unknown parameter  \( \mu \). All these five parameters are fixed by the (35), (36) and (38).

V. CONCLUSIONS

In this paper we analyzed the problem of on-line self calibration for autonomous vehicles from a theoretical point of view. We restricted our analysis to the odometry sensors and we gave an answer to the following two questions:

- does the system contain the necessary information to perform the self-calibration?
- what is the best trajectory in order to maximize the accuracy of the calibration?

Regarding the first question we considered several holo-nomic and non-holonomic mobile vehicles. We also analyzed the case when these systems contain the information to perform the simultaneous localization and odometry calibration (SLAC). Regarding the second question, we restricted our analysis to the case of holonomic vehicles. For this case, we derived second order differential equations which characterize the best solution (i.e. the solution maximizing the calibration accuracy).

REFERENCES


APPENDIX

We want to show that the gradients  \( dL^0 h, dL^1 h, dL^1_1 h, \) \( dL^3_1 h \) and \( dL^3_1_1 h \) are independent. The system is defined in (6). The observation  \( h \) is the distance  \( D \). We start our proof by computing the five Lie derivatives  \( L^0 h, L^1 h, L^1_1 h, L^1_1 h \) and \( L^1_1_1 h \). By using \( h = D \) and from the expression of  \( f_1 \) and  \( f_2 \) given in (7) we get:
\[ L^0 h = D \]
\[ L^1 h = A \cos \theta \]
\[ L^1_1 h = AB \cos \theta \]
\[ L^3_1 h = -AC \sin \theta + \frac{A^2 \sin^2 \theta}{D} \]
\[ L^3_1_1 h = -3 \frac{A^3}{D^2} \sin^2 \theta \cos \theta - AC^2 \cos \theta + 3 \frac{A^2 C}{D} \sin \theta \cos \theta \]

To compute the gradients we simply need to differentiate all the previous functions with respect to  \( \theta, A, B \) and  \( C \). We have for instance  \( dL^0 h = [1, 0, 0, 0, 0] \). To prove that the previous gradients are independent we stack them in a matrix and we compute its determinant. We obtain \( det = -A \cos \theta \left[ \frac{A^2 \sin \theta}{D} + 2A^2 C^2 \cos \theta \right] \) which is different from 0 except when  \( \theta = \frac{\pi}{2} + 2n\pi \) and/or  \( C = -\frac{A \sin \theta}{2D} \).